Decoupling Solution Moduli of Bigravity

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Abstract

A complete classification of exact solutions of ghost-free, massive bigravity is derived which enables the dynamical decoupling of the background, and the foreground metrics. The general decoupling solution space of the two metrics is constructed. Within this branch of the solution space the foreground metric theory becomes general relativity (GR) with an additional effective cosmological constant, and the background metric dynamics is governed by plain GR.
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1 Introduction

Recently, a ghost-free [1, 2] nonlinear massive gravity theory was constructed in [3, 4]. This theory is a nonlinear generalization of [5]. Independently later on, this ghost-free massive gravity with a flat reference metric was also extended to include a general background metric in [6, 7, 8]. A ghost-free

two-dynamical-metric theory, namely the bigravity as a covering theory of the massive gravity has also been proposed by introducing the dynamics for the background metric [9, 10, 11, 12].

In this paper, by referring to the simple observation already pointed out in [10] which leads to a dynamical decoupling of the background $[f]$, and the foreground $[g]$ metrics we will derive the general solutions $f(x^\mu) = F(g(x^\mu), x^\mu)$ which enable the two metrics to be solutions of two disjoint general relativity (GR) theories. This is possible if a portion of the effective energy-momentum tensor entering into the g-metric equations as a course of the interaction Lagrangian between the two metrics vanishes. As the same term also appears in the f-metric equations by being the only contribution, when it vanishes the sets of field equations of the two metrics completely decouple from each other yielding only an algebraic matrix equation which generates this picture. This matrix equation written for f, and g plays the role of a solution ansatz that leads us to a branch of the solution space generated by a Cartesian product of two GR's. This matrix ansatz equation will be at the center of our analysis. In the following, we will derive the general solutions of this cubic matrix equation when none of the parameters of the theory vanish. Thus, we will be able to give a complete description of the solution space $\Gamma[f, g]$ whose elements lead to this dynamical decoupling of the two metric sectors. We will also show that, the classification scheme of the analytically available solutions $\{f, g\}$ admits a similarity equivalence class structure.

In Section one, following a summary of the bigravity dynamics we will obtain the decoupling ansatz matrix equation we have mentioned above. Then, in the next section, we will derive the general solutions of this cubic matrix equation for generic constant coefficients. Since the coefficients in the actual equation are functions of the elementary symmetric polynomials of the solutions themselves a more refining analysis is needed. Therefore, later on, we will present a parametric derivation which enables us to construct not only the solutions of this involved matrix equation, but also their elementary symmetric polynomials as functions of the parameters of the theory. Subsequently, we will show that a subset of the generic solutions for constant coefficients must be omitted, when one insists on having an entire set of nonzero theory parameters. Besides, some of the generic solutions are forced to yield the same form when they are plugged into the actual matrix equation we have. In Section four, we will also present a formal definition of the decoupling solution space $\Gamma[f, g]$ of the bigravity theory. We will show

that, this space contains a major subset that is composed of analytically well-defined similarity equivalence classes of solutions. Finally, in Section five we will explicitly construct the proportional background solutions, and give an example of Friedmann-Lemaitre-Robertson-Walker (FLRW) on FLRW case.

2 The dynamical decoupling

The action for the ghost-free bimetric gravity [9, 10, 11, 12] for the foreground, g, and the background, f metrics in the presence of two types of matter can be given as

$$\begin{aligned} S = & -\frac{1}{16\pi G}\int dx^4\sqrt{-g}\Big[R^g + \Lambda^g - 2m^2\mathcal{L}_{int}(\sqrt{\Sigma})\Big] + S_M^g \\ & -\frac{\kappa}{16\pi G}\int dx^4\sqrt{-f}\Big[R^f + \Lambda^f\Big] + \epsilon S_M^f, \end{aligned} \tag{2.1}$$

where $R^g, R^f, \Lambda^g, \Lambda^f$ are the corresponding Ricci scalars, and the cosmological constants for the two metrics, respectively. S_M^g, S_M^f are the two different types of matter which independently couple to g, and f, respectively. The interaction Lagrangian of the two metrics is

$$\mathcal{L}_{int}(\sqrt{\Sigma}) = \beta_1 e_1(\sqrt{\Sigma}) + \beta_2 e_2(\sqrt{\Sigma}) + \beta_3 e_3(\sqrt{\Sigma}), \tag{2.2}$$

where $\{e_n\}$ are the elementary symmetric polynomials

$$\begin{aligned} e_1 &\equiv e_1(\sqrt{\Sigma}) = tr\sqrt{\Sigma}, \\ e_2 &\equiv e_2(\sqrt{\Sigma}) = \frac{1}{2}\big((tr\sqrt{\Sigma})^2 - tr(\sqrt{\Sigma})^2\big), \\ e_3 &\equiv e_3(\sqrt{\Sigma}) = \frac{1}{6}\big((tr\sqrt{\Sigma})^3 - 3\,tr\sqrt{\Sigma}\,tr(\sqrt{\Sigma})^2 + 2\,tr(\sqrt{\Sigma})^3\big), \end{aligned} \tag{2.3}$$

of the square-root-matrix

$$\sqrt{\Sigma} = \sqrt{g^{-1}f}. \tag{2.4}$$

Likewise in [12] the original interaction terms $\beta_0 e_0 = \beta_0$, and $\beta_4 e_4 = \beta_4 \det\sqrt{\Sigma}$ are trivially plugged into the cosmological constants Λ_g, and Λ_f, respectively. If we demand that Eq.(2.2) gives the Fierz-Pauli form in the weak-field limit then we must have [9]

$$\beta_1 + 2\beta_2 + \beta_3 = -1. \tag{2.5}$$

Varying Eq.(2.1) with respect to g gives the g-equation

$$R^g_{\mu\nu} - \frac{1}{2}R^g g_{\mu\nu} - \frac{1}{2}\Lambda^g g_{\mu\nu} - m^2\mathcal{T}^g_{\mu\nu} = 8\pi G T^g_{M\,\mu\nu}. \tag{2.6}$$

Also, variation with respect to f results in the f-equation

$$\kappa\big[R^f_{\mu\nu} - \frac{1}{2}R^f f_{\mu\nu} - \frac{1}{2}\Lambda^f f_{\mu\nu}\big] - m^2\mathcal{T}^f_{\mu\nu} = \epsilon 8\pi G T^f_{M\,\mu\nu}. \tag{2.7}$$

In these field equations the contributions coming from the interaction term that is given in Eq.(2.2) are the effective energy-momentum tensors

$$\mathcal{T}^g_{\mu\nu} = g_{\mu\rho}\tau^\rho{}_\nu - \mathcal{L}_{int} g_{\mu\nu}, \tag{2.8}$$

and

$$\mathcal{T}^f_{\mu\nu} = -\frac{\sqrt{-g}}{\sqrt{-f}} f_{\mu\rho}\tau^\rho{}_\nu, \tag{2.9}$$

respectively. Here $\{\tau^\rho{}_\nu\}$ are the elements of the matrix τ [12]

$$\tau = \beta_3(\sqrt{\Sigma})^3 - (\beta_2 + \beta_3 e_1)(\sqrt{\Sigma})^2 + (\beta_1 + \beta_2 e_1 + \beta_3 e_2)\sqrt{\Sigma}, \tag{2.10}$$

namely $\tau^\rho{}_\nu \equiv [\tau]^\rho{}_\nu$. Both of the effective energy-momentum tensors must be covariantly constant

$$\nabla^g_\mu(\mathcal{T}^g)^\mu{}_\nu = 0, \quad \nabla^f_\mu(\mathcal{T}^f)^\mu{}_\nu = 0. \tag{2.11}$$

If one of these constraints is satisfied then the other one is automatically satisfied [13, 14]. As discussed in [10] if one chooses

$$\tau = 0, \tag{2.12}$$

then dynamically the Eqs.(2.6), and (2.7) decouple from each other. In this case the first of the constraints (2.11) gives

$$\partial_\mu \mathcal{L}_{int} = 0, \tag{2.13}$$

and thus, as the solution we will take

$$\mathcal{L}_{int} = -\frac{1}{2}\tilde{\Lambda}. \tag{2.14}$$

Therefore, from Eqs.(2.8), (2.9) we have

$$\mathcal{T}^g_{\mu\nu} = \frac{1}{2}\tilde{\Lambda} g_{\mu\nu}, \quad \mathcal{T}^f_{\mu\nu} = 0. \tag{2.15}$$

Consequently, the g-equation Eq.(2.6) becomes the usual Einstein equations for g with an additional effective cosmological constant $\tilde{\Lambda}$, whereas the dynamically-disjoint f-equation Eq.(2.7) reduces to be coupling-constant-modified Einstein equations for f. The rest of our analysis will be devoted to find the general solutions of the matrix equation[1]

$$\sqrt{\Sigma}\Big(A(\sqrt{\Sigma})^2 + B(\sqrt{\Sigma}) + C\mathbf{1}_4\Big) = 0, \tag{2.16}$$

where

$$\begin{aligned} A &= -\beta_3, \\ B &= \beta_2 + \beta_3 e_1, \\ C &= -\beta_1 - \beta_2 e_1 - \beta_3 e_2, \end{aligned} \tag{2.17}$$

which constitute the effective solution space $\Gamma[g, f]$ of the ghost-free bigravity action (2.1) that enables the above-mentioned dynamical decoupling for the foreground, and the background metrics.

3 The structure of the solution space Γ

Now, let us consider the matrix equation

$$AX^3 + BX^2 + CX = X(AX^2 + BX + C\mathbf{1}_4) = X(X - \lambda_1\mathbf{1}_4)(X - \lambda_2\mathbf{1}_4) = 0, \tag{3.1}$$

for a 4×4 matrix function $X(x^\mu)$. For the following analysis we will disregard the solutions which require either of the β-coefficients to be zero. The characteristic polynomial of any 4×4 matrix X would be the degree-four polynomial

$$P_X(t) = det(t\mathbf{1}_4 - X), \tag{3.2}$$

whose four roots are the eigenvalues of X. We should observe first that if X is a solution of Eq.(3.1) then for any invertible matrix function $P(x^\mu)$, $P^{-1}XP$

[1] We will prefer to work with the negative of the Eq.(2.12) in accordance with the massive gravity formalism [15, 16], and for future relevance.

is also a solution. Therefore, in order to find the general solutions of Eq.(3.1) it would be sufficient to classify the Jordan canonical forms satisfying (3.1). Our main objective next, will be the classification of the similarity equivalence classes of solutions with respect to their minimum polynomials $m(X)$. The roots of $m(X)$ are the same with the eigenvalues of the various equivalence classes of matrices having that minimum polynomial with differing multiplicities of course. Since Eq.(3.1) is a degree-three polynomial equation when it is factorized, its various factors with degrees smaller than or equal to three will define the minimum polynomials of its 4×4 matrix function solutions. In other words, the solutions can be classified with respect to these minimum polynomial factors.

3.1 The algebraic structure

In the following classification, we will identify the entire set of similarity equivalence classes of solutions satisfying Eq.(3.1) by simply taking the coefficients in Eq.(3.1) to be constants. We will group the solutions with respect to their minimum polynomials and it will be sufficient to determine the Jordan canonical form spectrum of each minimum polynomial which corresponds to some combination of the factors in Eq.(3.1).

1. *Solutions with* $m(X) = X$

   Since now, the minimum polynomial has no repeated roots and its unique root is zero these solutions are diagonalizable with zero eigenvalues. They are the trivial solutions $X = 0$. These solutions would demand at least one of the metrics to be zero via Eq(2.4) as metrics are invertible matrices.

   Next, we will classify the solutions of Eq.(3.1) with respect to the root structure of the quadratic factor.

   - **The cases when $B^2 - 4AC > 0$:**

   In this case, the factor $AX^2 + BX + C\mathbf{1}_4$ has two distinct real roots λ_1, λ_2.

2. *Solutions with* $m(X) = X - \lambda_{1,2}\mathbf{1}_4$

   Now, there exist diagonal solutions as the minimum polynomials have no repeated roots and they are linear. These solutions are

$$U_1 = \lambda_1 \mathbf{1}_4, \qquad U_2 = \lambda_2 \mathbf{1}_4, \tag{3.3}$$

with minimum polynomials $m(X) = X - \lambda_1 \mathbf{1}_4,\ m(X) = X - \lambda_2 \mathbf{1}_4$, respectively.

3. *Solutions with* $m(X) = AX^2 + BX + C\mathbf{1}_4$

   The matrices with a minimum polynomial of the form $m(X) = AX^2 + BX + C\mathbf{1}_4$ are diagonalizable when the roots are distinct like our case here. Thus, by considering all the possible multiplicities of the eigenvalues which are the same with these distinct roots λ_1, λ_2 the Jordan forms corresponding to this minimum polynomial become

$$U_3 = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}, U_4 = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix},$$
$$U_5 = \begin{pmatrix} \lambda_2 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}. \tag{3.4}$$

4. *Solutions with* $m(X) = X(X - \lambda_1 \mathbf{1}_4)$

   In this case, the roots of the minimum polynomial are $\{0, \lambda_1\}$. Again, the matrices with this minimum polynomial are diagonalizable. The possible Jordan forms with $\{0, \lambda_1\}$ eigenvalues read

$$U_6 = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, U_7 = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, U_8 = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.5}$$

5. *Solutions with* $m(X) = X(X - \lambda_2 \mathbf{1}_4)$

   Similarly, the diagonalizable solutions with eigenvalues $\{0, \lambda_2\}$ as the roots of the minimum polynomial have the following possible Jordan forms

$$U_9 = \begin{pmatrix} \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, U_{10} = \begin{pmatrix} \lambda_2 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, U_{11} = \begin{pmatrix} \lambda_2 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.6}$$

6. *Solutions with* $m(X) = X(X - \lambda_1 \mathbf{1}_4)(X - \lambda_2 \mathbf{1}_4)$

   Now, the roots of the minimum polynomial are $\{0, \lambda_1, \lambda_2\}$. They are distinct, and this states that the solutions whose minimum polynomial is of this form must be again diagonalizable. The possible Jordan forms constructed from these eigenvalues are

$$U_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}, U_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix},$$

$$U_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}. \tag{3.7}$$

- **The cases when** $B^2 - 4AC = 0$ **:**

For these cases, the factor $AX^2 + BX + C\mathbf{1}_4$ has a repeated real root which we will call λ'.

7. *Solutions with* $m(X) = X - \lambda' \mathbf{1}_4$

   This solution, is another diagonal one due to its minimum polynomial. It is

$$V_0 = \lambda' \mathbf{1}_4. \tag{3.8}$$

8. *Solutions with* $m(X) = (X - \lambda' \mathbf{1}_4)^2$

   Since now, the minimum polynomial has repeated roots the matrices with such a minimum polynomial are nondiagonalizable. Their eigenvalues must be λ' with multiplicity four. Thus the primary block is four dimensional and since the multiplicity of the roots of $m(X)$ is two the maximum dimension of the secondary block must be two. The possible Jordan forms are

$$V_1 = \begin{pmatrix} \lambda' & 1 & 0 & 0 \\ 0 & \lambda' & 0 & 0 \\ 0 & 0 & \lambda' & 0 \\ 0 & 0 & 0 & \lambda' \end{pmatrix}, \quad V_2 = \begin{pmatrix} \lambda' & 0 & 0 & 0 \\ 0 & \lambda' & 1 & 0 \\ 0 & 0 & \lambda' & 0 \\ 0 & 0 & 0 & \lambda' \end{pmatrix},$$
$$V_3 = \begin{pmatrix} \lambda' & 0 & 0 & 0 \\ 0 & \lambda' & 0 & 0 \\ 0 & 0 & \lambda' & 1 \\ 0 & 0 & 0 & \lambda' \end{pmatrix}, \quad V_4 = \begin{pmatrix} \lambda' & 1 & 0 & 0 \\ 0 & \lambda' & 0 & 0 \\ 0 & 0 & \lambda' & 1 \\ 0 & 0 & 0 & \lambda' \end{pmatrix}. \tag{3.9}$$

We should remark that, it would be enough to take either of $\{V_1, V_2, V_3\}$ however, to be on the safe side we include all of them.

9. *Solutions with* $m(X) = X(X - \lambda'\mathbf{1}_4)^2$

   The roots of the minimum polynomial in this class of solutions are $\{0, \lambda', \lambda'\}$. Once more, $m(X)$ has repeated roots so the matrices with this minimum polynomial are nondiagonalizable. The dimension of the secondary blocks for the eigenvalues λ', and 0 are two, and one, respectively. The possible Jordan canonical forms are

$$V_5 = \begin{pmatrix} \lambda' & 1 & 0 & 0 \\ 0 & \lambda' & 0 & 0 \\ 0 & 0 & \lambda' & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, V_6 = \begin{pmatrix} \lambda' & 0 & 0 & 0 \\ 0 & \lambda' & 1 & 0 \\ 0 & 0 & \lambda' & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$V_7 = \begin{pmatrix} \lambda' & 1 & 0 & 0 \\ 0 & \lambda' & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.10}$$

   Like the previous case, we include both V_5, and V_6 in spite of the fact that one would be enough to generate the similarity class of solutions.

10. *Solutions with* $m(X) = X(X - \lambda'\mathbf{1}_4)$

   There are no repeated roots of $m(X)$ for these solutions. Therefore, the matrices with this minimum polynomial are diagonalizable. The

possible Jordan normal forms with the eigenvalues $\{0, \lambda'\}$ are

$$V_8 = \begin{pmatrix} \lambda' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, V_9 = \begin{pmatrix} \lambda' & 0 & 0 & 0 \\ 0 & \lambda' & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$V_{10} = \begin{pmatrix} \lambda' & 0 & 0 & 0 \\ 0 & \lambda' & 0 & 0 \\ 0 & 0 & \lambda' & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.11}$$

- **The cases when $B^2 - 4AC < 0$:**

For this class of solutions, the factor $AX^2 + BX + C\mathbf{1}_4$ has complex roots which we will call λ, λ^*. In this respect for such cases, the minimum polynomial of the similarity equivalence classes of real solutions must include the quadratic factor as a whole since for real matrices if λ is an eigenvalue λ^* must also be an eigenvalue, and vice versa. For this reason, there are two possible minimum polynomials and the corresponding solutions are nondiagonalizable.

11. *Solutions with* $m(X) = (X - \lambda\mathbf{1}_4)(X - \lambda^*\mathbf{1}_4)$

The eigenvalues of the matrices with this $m(X)$ are $\{\lambda, \lambda^*\}$. If we define the real and the imaginary parts of $\lambda = R + I\mathbf{i}$, then the unique Jordan canonical form for the real solutions with this minimum polynomial becomes

$$Y = \begin{pmatrix} R & I & 0 & 0 \\ -I & R & 0 & 0 \\ 0 & 0 & R & I \\ 0 & 0 & -I & R \end{pmatrix}. \tag{3.12}$$

12. *Solutions with* $m(X) = X(X - \lambda\mathbf{1}_4)(X - \lambda^*\mathbf{1}_4)$

The eigenvalues of the solutions of this equivalence class are $\{0, \lambda, \lambda^*\}$. The only possible Jordan normal form for the real matrices satisfying $m(X) = X(X - \lambda\mathbf{1}_4)(X - \lambda^*\mathbf{1}_4) = 0$ is of the form

$$Y' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & R & I \\ 0 & 0 & -I & R \end{pmatrix}. \tag{3.13}$$

For the purpose of achieving the most general construction of the solution space we should also consider the cases when the coefficients in Eq.(3.1) vanish. We will disregard the cases when $A = 0$ as they correspond to choosing $\beta_3 = 0$.

- **<u>The cases when $B = 0$:</u>**

If we assume that $C/A < 0$ [2] then the quadratic factor in Eq.(3.1) can be written as $(X - \tilde{\lambda})(X + \tilde{\lambda})$ where $\tilde{\lambda} = \sqrt{-C/A}$.

13. *Solutions with* $m(X) = X \pm \tilde{\lambda}\mathbf{1}_4$

    There exist diagonal solutions. They are

$$K_1 = \tilde{\lambda}\mathbf{1}_4, \qquad K_2 = -\tilde{\lambda}\mathbf{1}_4, \tag{3.14}$$

    with the minimum polynomials $m(X) = X - \tilde{\lambda}\mathbf{1}_4$, and $m(X) = X + \tilde{\lambda}\mathbf{1}_4$, respectively.

14. *Solutions with* $m(X) = AX^2 + C\mathbf{1}_4$

    The roots of the minimum polynomial thus, the eigenvalues in this case are $\{\tilde{\lambda}, -\tilde{\lambda}\}$. They are distinct hence, the matrices with this minimum polynomial are diagonalizable. The possible Jordan forms are

$$K_3 = \begin{pmatrix} \tilde{\lambda} & 0 & 0 & 0 \\ 0 & \tilde{\lambda} & 0 & 0 \\ 0 & 0 & \tilde{\lambda} & 0 \\ 0 & 0 & 0 & -\tilde{\lambda} \end{pmatrix}, K_4 = \begin{pmatrix} \tilde{\lambda} & 0 & 0 & 0 \\ 0 & \tilde{\lambda} & 0 & 0 \\ 0 & 0 & -\tilde{\lambda} & 0 \\ 0 & 0 & 0 & -\tilde{\lambda} \end{pmatrix},$$

$$K_5 = \begin{pmatrix} \tilde{\lambda} & 0 & 0 & 0 \\ 0 & -\tilde{\lambda} & 0 & 0 \\ 0 & 0 & -\tilde{\lambda} & 0 \\ 0 & 0 & 0 & -\tilde{\lambda} \end{pmatrix}. \tag{3.15}$$

15. *Solutions with* $m(X) = X(X - \tilde{\lambda}\mathbf{1}_4)$

    The roots of the minimum polynomial and the eigenvalues of the solutions in this case are $\{0, \tilde{\lambda}\}$. They are not repeated so, the corresponding matrices are diagonalizable. The Jordan canonical forms

[2] We will comment on the case when $C/A > 0$ in the next subsection.

with various eigenvalue multiplicities are

$$K_6 = \begin{pmatrix} \tilde{\lambda} & 0 & 0 & 0 \\ 0 & \tilde{\lambda} & 0 & 0 \\ 0 & 0 & \tilde{\lambda} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_7 = \begin{pmatrix} \tilde{\lambda} & 0 & 0 & 0 \\ 0 & \tilde{\lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$K_8 = \begin{pmatrix} \tilde{\lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.16}$$

16. *Solutions with* $m(X) = X(X + \tilde{\lambda}\mathbf{1}_4)$

    Now, the roots of the minimum polynomial and the eigenvalues of the solutions become $\{0, -\tilde{\lambda}\}$. Again, they are distinct, the corresponding matrices are diagonalizable. The possible Jordan canonical forms with various eigenvalue multiplicities are

$$K_9 = \begin{pmatrix} -\tilde{\lambda} & 0 & 0 & 0 \\ 0 & -\tilde{\lambda} & 0 & 0 \\ 0 & 0 & -\tilde{\lambda} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_{10} = \begin{pmatrix} -\tilde{\lambda} & 0 & 0 & 0 \\ 0 & -\tilde{\lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$K_{11} = \begin{pmatrix} -\tilde{\lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.17}$$

17. *Solutions with* $m(X) = X(AX^2 + C\mathbf{1}_4)$

    In this case, the roots of the minimum polynomial are $\{0, \tilde{\lambda}, -\tilde{\lambda}\}$. They are not repeated, hence, the corresponding matrices are diagonalizable. The possible Jordan canonical forms constructed with various multi-

plicities of these eigenvalues can be listed as

$$K_{12} = \begin{pmatrix} \tilde{\lambda} & 0 & 0 & 0 \\ 0 & -\tilde{\lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_{13} = \begin{pmatrix} \tilde{\lambda} & 0 & 0 & 0 \\ 0 & \tilde{\lambda} & 0 & 0 \\ 0 & 0 & -\tilde{\lambda} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K_{14} = \begin{pmatrix} \tilde{\lambda} & 0 & 0 & 0 \\ 0 & -\tilde{\lambda} & 0 & 0 \\ 0 & 0 & -\tilde{\lambda} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.18}$$

- **<u>The cases when $C = 0$:</u>**

18. *Solutions with* $m(X) = AX + B\mathbf{1}_4$

    Since, the minimum polynomial is linear this case is a diagonal solution. It is

$$X_1 = -\frac{B}{A}\mathbf{1}_4. \tag{3.19}$$

19. *Solutions with* $m(X) = X(AX + B\mathbf{1}_4)$

    The roots of $m(X)$ now, become $\{0, -B/A\}$. They are distinct thus, the matrices with this minimum polynomial are diagonalizable with eigenvalues $\{0, -B/A\}$. The possible Jordan normal forms are

$$X_2 = \begin{pmatrix} -B/A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} -B/A & 0 & 0 & 0 \\ 0 & -B/A & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_4 = \begin{pmatrix} -B/A & 0 & 0 & 0 \\ 0 & -B/A & 0 & 0 \\ 0 & 0 & -B/A & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.20}$$

20. *Solutions with* $m(X) = X^2(AX + B\mathbf{1}_4)$

For this branch of solutions, the minimum polynomial $m(X)$ has repeated roots which are $\{0, 0, -B/A\}$. Hence, the matrices with this minimum polynomial are nondiagonalizable. In the relevant Jordan canonical forms, the dimension of the secondary block corresponding to the zero eigenvalue is two, and for the $-B/A$ eigenvalue it is one. The possible Jordan forms are

$$X_5 = \begin{pmatrix} -B/A & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, X_6 = \begin{pmatrix} -B/A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$X_7 = \begin{pmatrix} -B/A & 0 & 0 & 0 \\ 0 & -B/A & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.21}$$

3.2 The parametric structure

In the previous subsection, we have derived and classified all the representatives of the similarity equivalence classes of the solutions of Eq.(3.1). The reader may verify that the matrices listed for various conditions satisfy Eq.(3.1) by direct substitution for unspecified constant coefficients under these conditions. Therefore, all the matrix functions constructed from these representatives by similarity transformations via an invertible matrix function $P(x^\mu)$ also satisfy Eq.(3.1). However, in Eq.(2.16) the coefficients of the matrix equation for $\sqrt{\Sigma}$ are not numerical constants, on the contrary, they are functions of the elementary symmetric polynomials e_1, e_2 of $\sqrt{\Sigma}$ which turn this matrix equation into a very nontrivial one. In this respect, our task is not over and we have to furthermore solve the entries of the matrices listed in the previous subsection explicitly, so that they satisfy this nontrivial matrix equation Eq.(2.16).

- **The solutions; $U_{1,2,3,4,5}, V_{0,1,2,3,4}, Y$:**

In this case, the solutions satisfy

$$AX^2 + BX + C\mathbf{1}_4 = 0. \tag{3.22}$$

By taking the trace of this equation, and using Eqs.(2.3), and (2.17) we get

$$3\beta_2 e_1 + 2\beta_3 e_2 + 4\beta_1 = 0. \tag{3.23}$$

For the cases which satisfy $B^2 - 4AC > 0$ the roots of Eq.(3.22) are

$$\lambda_1 = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \quad \lambda_2 = \frac{-B - \sqrt{B^2 - 4AC}}{2A}. \tag{3.24}$$

From Eqs.(3.3), and (3.4) we have

$$e_1 = tr(U_i) = n\lambda_1 + m\lambda_2, \tag{3.25}$$

for U_1, U_2, U_3, U_4, U_5 the (n, m) values are $(4, 0), (0, 4), (2, 2), (3, 1), (1, 3)$, respectively. When $m - n \neq \pm 2$ namely, for U_1, U_2, U_3 using Eqs.(3.23), and (3.24) in Eq.(3.25) yields the quadratic equation

$$a(e_1)^2 + be_1 + c = 0, \tag{3.26}$$

with

$$\begin{aligned}
&a = 2(n-1)(m-1)(\beta_3)^2, \\
&b = -\big((m-n)^2 + 2(m+n-2mn)\big)\beta_2\beta_3, \\
&c = 2mn(\beta_2)^2 - 2(n-m)^2\beta_1\beta_3.
\end{aligned} \tag{3.27}$$

Therefore, provided

$$b^2 - 4ac \geq 0, \tag{3.28}$$

e_1 becomes

$$e_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{3.29}$$

and from Eq.(3.23) we have

$$e_2 = -\frac{3\beta_2}{2\beta_3}\left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right) - \frac{2\beta_1}{\beta_3}, \tag{3.30}$$

For U_3, $b^2 - 4ac = 0$ thus, the condition (3.28) is automatically satisfied. For U_1, and U_2 it becomes

$$3(\beta_2)^2 - 4\beta_1\beta_3 \geq 0. \tag{3.31}$$

Furthermore, we can now explicitly write the components of U_1, U_2, U_3 as

$$\lambda_1 = \frac{\beta_2 + \beta_3\big(\frac{-b\pm\sqrt{b^2-4ac}}{2a}\big) - \sqrt{\Delta}}{2\beta_3}, \quad \lambda_2 = \frac{\beta_2 + \beta_3\big(\frac{-b\pm\sqrt{b^2-4ac}}{2a}\big) + \sqrt{\Delta}}{2\beta_3}, \tag{3.32}$$

where

$$\Delta = (\beta_3)^2\Big(\frac{-b \pm \sqrt{b^2-4ac}}{2a}\Big)^2 + 4\beta_2\beta_3\Big(\frac{-b \pm \sqrt{b^2-4ac}}{2a}\Big) + (\beta_2)^2 + 4\beta_1\beta_3. \tag{3.33}$$

In this expression, by substituting a, b, c, from Eq(3.27) the restriction $\Delta = B^2 - 4AC > 0$ that defines these solutions takes the form

$$-3(\beta_2)^2 + 4\beta_1\beta_3 > 0, \tag{3.34}$$

for U_3, and it reads

$$3(\beta_2)^2 - 4\beta_1\beta_3 > 0, \tag{3.35}$$

for U_1, and U_2. The condition (3.35) is stronger than (3.31) hence, it must be taken as the defining constraint on the parameter space for the solutions U_1, U_2 to exist. Therefore, when the coefficients $\{\beta_i\}$ satisfy Eq.(3.34) the similarity equivalence class of U_3 (upon the substitution of the matrix elements via (3.32)) are solutions of Eq.(2.16). Also, when the coefficients $\{\beta_i\}$ satisfy Eq.(3.35) the similarity equivalence classes of U_1, and U_2 again, by using the relevant matrix entries from (3.32) are solutions of Eq.(2.16). On the other hand, when $m - n = \pm 2$, namely, for the cases U_4, U_5 Eqs.(3.23), (3.24), and (3.25) lead us to the condition

$$3(\beta_2)^2 - 4\beta_1\beta_3 = 0. \tag{3.36}$$

By using this condition one can show that we always have

$$\Delta_{m-n=\pm 2} = B^2 - 4AC = (2\beta_2 + \beta_3 e_1)^2 > 0. \tag{3.37}$$

Thus, for these cases $\Delta > 0$ is always satisfied provided (3.36) holds. The roots of the minimum polynomial now become

$$\lambda_1 = \frac{\beta_2 + \beta_3 e_1 - (\pm(2\beta_2 + \beta_3 e_1))}{2\beta_3}, \quad \lambda_2 = \frac{\beta_2 + \beta_3 e_1 + (\pm(2\beta_2 + \beta_3 e_1))}{2\beta_3}. \tag{3.38}$$

By using these in U_4, and U_5 the consistency condition (3.25) demands that the signs in (3.38) must be chosen such that

$$U_4 = \begin{pmatrix} -\frac{\beta_2}{2\beta_3} & 0 & 0 & 0 \\ 0 & -\frac{\beta_2}{2\beta_3} & 0 & 0 \\ 0 & 0 & -\frac{\beta_2}{2\beta_3} & 0 \\ 0 & 0 & 0 & \frac{3\beta_2+2\beta_3 e_1}{2\beta_3} \end{pmatrix},$$
$$U_5 = \begin{pmatrix} \frac{3\beta_2+2\beta_3 e_1}{2\beta_3} & 0 & 0 & 0 \\ 0 & -\frac{\beta_2}{2\beta_3} & 0 & 0 \\ 0 & 0 & -\frac{\beta_2}{2\beta_3} & 0 \\ 0 & 0 & 0 & -\frac{\beta_2}{2\beta_3} \end{pmatrix}, \tag{3.39}$$

which are similar to each other thus, we will take the representative of the equivalence class as $U \equiv U_4$. We remark that, in this case the formulation does not specify e_1 and it remains as a free parameter. It can take any value except $-2\beta_2/\beta_3$ which would violate (3.37). Therefore, the solution $U \equiv U_4$, is parametrized by a free trace parameter e_1 as in Eq.(3.39), and from (3.23) we have

$$e_2 = -\frac{3\beta_2}{2\beta_3}e_1 - \frac{2\beta_1}{\beta_3}. \tag{3.40}$$

Among the solutions which satisfy Eq.(3.22); when $B^2 - 4AC = 0$ we have the ones V_0, V_1, V_2, V_3, V_4, and when $B^2 - 4AC < 0$ we have the solution Y. For all of these cases taking the trace of these matrices leads us to

$$e_1 = -\frac{2\beta_2}{\beta_3}. \tag{3.41}$$

From Eq.(3.23) we get

$$e_2 = \frac{3(\beta_2)^2}{(\beta_3)^2} - \frac{2\beta_1}{\beta_3}. \tag{3.42}$$

From these relations for V_0, V_1, V_2, V_3, V_4 via the definitions in Eq(2.17) we have

$$\lambda' = -\frac{B}{2A} = -\frac{\beta_2}{2\beta_3}, \tag{3.43}$$

and the condition $B^2 - 4AC = 0$ which allows these solutions to exist becomes

$$3(\beta_2)^2 - 4\beta_1\beta_3 = 0. \tag{3.44}$$

On the other hand, when $B^2 - 4AC < 0$ is valid Eq.(3.22) has two complex roots $\lambda = R + Ii$, and $\lambda^* = R - Ii$, here

$$R = -\frac{\beta_2}{2\beta_3}, \quad I = -\frac{\sqrt{3(\beta_2)^2 - 4\beta_1\beta_3}}{2\beta_3}, \tag{3.45}$$

where we have used Eqs.(2.17), (3.41), and (3.42). From the same equations the condition $B^2 - 4AC < 0$ which allows the solution Y to exist now, becomes

$$-3(\beta_2)^2 + 4\beta_1\beta_3 < 0. \tag{3.46}$$

- **The solutions; $U_{6,\ldots,14}, V_{5,\ldots,10}, Y'$:**

For the $U-$matrices

$$e_1 = n\lambda_1 + m\lambda_2 = -(n+m)\frac{B}{2A} + (n-m)\frac{\sqrt{B^2 - 4AC}}{2A}, \tag{3.47}$$

where $(n,m) = (1,0),(2,0),(3,0),(0,1),(0,2),(0,3),(2,1),(1,1),(1,2)$ for $U_6, U_7, U_8, U_9, U_{10}, U_{11}, U_{12}, U_{13}, U_{14}$, respectively. We also have

$$e_2 = \frac{1}{2}((e_1)^2 - tr(U_i)^2). \tag{3.48}$$

Now, from this relation by using $tr(U_i)^2 = n(\lambda_1)^2 + m(\lambda_2)^2$, and the fact that λ_1, λ_2 are the roots of $Ax^2 + Bx + C = 0$ we obtain

$$e_2 = \frac{1}{(2-n-m)}\Big((n+m-1)\frac{\beta_2}{\beta_3}e_1 + (n+m)\frac{\beta_1}{\beta_3}\Big), \tag{3.49}$$

when $n + m \neq 2$. Again, for the cases $n + m \neq 2$ substituting Eqs. (2.17), and (3.49) into Eq.(3.47) after some algebra leads us to the equation

$$a'(e_1)^2 + b'e_1 + c' = 0, \tag{3.50}$$

where

$$\begin{aligned}
a' &= (2-n-m)(1-m)(1-n)(\beta_3)^2, \\
b' &= -\big((2-n-m)(1-m)n + (2-n-m)(1-n)m - (n-m)^2\big)\beta_2\beta_3, \\
c' &= mn(2-n-m)(\beta_2)^2 + 2(n-m)^2\beta_1\beta_3.
\end{aligned} \tag{3.51}$$

If the condition
$$b'^2 - 4a'c' \geq 0, \tag{3.52}$$
is satisfied then e_1 becomes
$$e_1 = \frac{-b' \pm \sqrt{b'^2 - 4a'c'}}{2a'}, \tag{3.53}$$
and via Eq.(3.49) we get
$$e_2 = \frac{1}{(2-n-m)}\left((n+m-1)\frac{\beta_2}{\beta_3}\Big(\frac{-b' \pm \sqrt{b'^2 - 4a'c'}}{2a'}\Big) + (n+m)\frac{\beta_1}{\beta_3}\right), \tag{3.54}$$
which is now written solely in terms of the parameters $\{\beta_i\}$'s. We can also write the components of the U-matrices as
$$\lambda_{1,2} = -\frac{1}{2\beta_3}\left(- \Big(\beta_2 + \beta_3\Big(\frac{-b' \pm \sqrt{b'^2 - 4a'c'}}{2a'}\Big)\Big) \pm \sqrt{\Delta}\right), \tag{3.55}$$
where
$$\begin{aligned} \Delta &= B^2 - 4AC \\ &= \left(\beta_2 + \beta_3\Big(\frac{-b' \pm \sqrt{b'^2 - 4a'c'}}{2a'}\Big)\right)^2 + 4\beta_3\Bigg[- \beta_1 - \beta_2\Big(\frac{-b' \pm \sqrt{b'^2 - 4a'c'}}{2a'}\Big) \\ &\quad - \frac{1}{(2-n-m)}\left((n+m-1)\beta_2\Big(\frac{-b' \pm \sqrt{b'^2 - 4a'c'}}{2a'}\Big) + (n+m)\beta_1\right)\Bigg]. \end{aligned} \tag{3.56}$$
For the existence of these solutions we must have $\Delta > 0$, together with the condition (3.52). In spite of the fact that, Eq.(3.50) is generally valid for all of the matrices; $U_{6,8,9,11,12,14}$ we must be cautious since for some of these solutions it reduces to a redundant form, or it is trivially satisfied. We will consider such cases one by one. In particular, for U_6, and U_9 (3.52) is satisfied directly as for these cases $b'^2 - 4a'c' = 0$. However, for these cases Eq.(3.50) holds only if $\beta_1\beta_3 = 0$, thus, we will exclude the solutions U_6, U_9. For U_8, U_{11} (3.52) becomes $(\beta_2)^2 - \beta_1\beta_3 \geq 0$, and $\Delta > 0$ leads to
$$(\beta_2)^2 - \beta_1\beta_3 > 0, \tag{3.57}$$
which is stronger than (3.52). Therefore, for the existence of the solutions U_8, U_{11}, Eq.(3.57) is the only condition. For the matrices U_{12}, U_{14} we have

$b'^2 - 4a'c' = 0$ hence, Eq.(3.52) is satisfied. Also, now Eq.(3.50) turns into the condition

$$-(\beta_2)^2 + \beta_1\beta_3 = 0, \tag{3.58}$$

and by using this condition for these cases, we find that

$$\Delta = (3\beta_2 + \beta_3 e_1)^2 > 0. \tag{3.59}$$

We observe that, for U_{12}, U_{14} if (3.58) is satisfied we always have $\Delta > 0$, this fact makes Eq.(3.58) the unique condition for the existence of these solutions. For the cases U_{12}, U_{14} the roots of the minimum polynomial are

$$\lambda_1 = \frac{\beta_2 + \beta_3 e_1 - (\pm(3\beta_2 + \beta_3 e_1))}{2\beta_3}, \quad \lambda_2 = \frac{\beta_2 + \beta_3 e_1 + (\pm(3\beta_2 + \beta_3 e_1))}{2\beta_3}. \tag{3.60}$$

When we substitute them in the matrices U_{12}, and U_{14} the consistency condition (3.47) constrains us to choose the signs in Eq.(3.60) in such a way that

$$U_{12} = U_{14} \equiv U' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{\beta_2}{\beta_3} & 0 & 0 \\ 0 & 0 & -\frac{\beta_2}{\beta_3} & 0 \\ 0 & 0 & 0 & \frac{2\beta_2+\beta_3 e_1}{\beta_3} \end{pmatrix}, \tag{3.61}$$

where e_1 remains to be a free trace parameter, as the matrix (3.61) trivially satisfies Eq.(3.47). It can take any value except $-3\beta_2/\beta_3$ which would violate (3.59). For the solution U', which is parametrized by e_1 from Eq.(3.49) we have

$$e_2 = -\frac{2\beta_2}{\beta_3} e_1 - \frac{3\beta_1}{\beta_3}. \tag{3.62}$$

On the other hand, if we turn our attention to the cases when $n + m = 2$, namely, the cases; U_7, U_{10}, U_{13} again, via $tr(U_i)^2 = n(\lambda_1)^2 + m(\lambda_2)^2$, and the fact that λ_1, λ_2 satisfy $Ax^2 + Bx + C = 0$, Eq.(3.48) gives

$$e_1 = -\frac{2\beta_1}{\beta_2}. \tag{3.63}$$

For U_{13}, we have $e_1 = \lambda_1 + \lambda_2$ however, this relation together with Eq.(3.63) result in the constraint $\beta_2 = 0$. From Eq.(3.63) in this case we must also have $\beta_1 = 0$. Therefore, we will disregard U_{13}. For the matrices U_7, U_{10},

$$e_1 = 2\lambda_{1,2} = 2\left(\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}\right), \tag{3.64}$$

where + corresponds to U_7, and $-$ to U_{10}, respectively. By referring to the definitions in Eq.(2.17), and by using Eq.(3.63) in Eq.(3.64) we find that

$$e_2 = \frac{(\beta_1)^2}{(\beta_2)^2}. \tag{3.65}$$

For the solutions U_7, U_{10}, upon the substitution of e_1, e_2 from Eqs.(3.63), and (3.65) the defining condition $B^2 - 4AC > 0$ becomes

$$(\beta_2)^2 > 0, \tag{3.66}$$

which is automatically satisfied. For these cases, by referring to the Eqs. (2.17), (3.63), and (3.65) the computation of the matrix entries reads

$$\lambda_{1,2} = \frac{(\beta_2)^2 - 2\beta_1\beta_3 \mp (\pm(\beta_2)^2)}{2\beta_3\beta_2}. \tag{3.67}$$

By substituting these in the matrices U_7, and U_{10}, the consistency condition (3.63) denotes that the signs in (3.67) must be chosen such that

$$U_7 = U_{10} \equiv U'' = \begin{pmatrix} -\frac{\beta_1}{\beta_2} & 0 & 0 & 0 \\ 0 & -\frac{\beta_1}{\beta_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.68}$$

Let us now focus on the solutions $V_{5,6,7,8,9,10}$. They have

$$e_1 = trV_i = n\lambda', \quad tr(V_i)^2 = n(\lambda')^2 = \frac{(e_1)^2}{n}, \tag{3.69}$$

where for V_5, V_6, V_{10} $n = 3$, for V_7, V_9 $n = 2$, and for V_8 $n = 1$. Furthermore,

$$e_2 = \frac{1}{2}\big((e_1)^2 - tr(V_i)^2\big) = \frac{(n-1)}{2n}(e_1)^2. \tag{3.70}$$

Referring to the definitions of the coefficients in Eq.(2.17) once more, explicitly we have

$$\lambda' = -\frac{B}{2A} = \frac{\beta_2 + \beta_3 e_1}{2\beta_3}. \tag{3.71}$$

Using this relation in Eq.(3.69) gives

$$e_1 = \frac{n\beta_2}{(2-n)\beta_3}, \tag{3.72}$$

when $n \neq 2$. If we consider the matrices V_7, V_9 for which $n = 2$ this computation leads to $\beta_2 = 0$, hence, we also disregard these solutions. Next, substituting Eq.(3.72) in Eq.(3.70) we also obtain e_2 in terms of the β_i-parameters as

$$e_2 = \frac{n(n-1)(\beta_2)^2}{2(2-n)^2(\beta_3)^2}. \tag{3.73}$$

Finally, by substituting Eq.(3.72) into Eq.(3.71) we can also calculate explicitly the matrix entries as

$$\lambda' = \frac{\beta_2}{(2-n)\beta_3}. \tag{3.74}$$

The condition $B^2 - 4AC = 0$, needed for the existence of the solutions $V_{5,6,7,8,9,10}$ reads

$$\left[\frac{1}{(2-n)^2} - \frac{n}{2-n} - \frac{n(n-1)}{2(n-2)^2}\right](\beta_2)^2 - \beta_1\beta_3 = 0, \tag{3.75}$$

where we have used Eqs. (2.17), (3.72), and (3.73). In particular, for the case V_8; $n = 1$, this condition gives $\beta_1\beta_3 = 0$ thus, we also disregard V_8. For the remaining cases $V_{5,6,10}$, since $n = 3$ Eq.(3.75) takes the form

$$(\beta_2)^2 - \beta_1\beta_3 = 0. \tag{3.76}$$

Our final case in this set of solutions is Y' which exists when $B^2 - 4AC < 0$. Starting from its definition in Eq.(3.13) we find that

$$e_1 = trY' = 2R, \tag{3.77}$$

where

$$R = -\frac{B}{2A}. \tag{3.78}$$

By substituting the definitions of A, B from Eq.(2.17) into Eq.(3.77) we get $\beta_2 = 0$. Therefore, Y' will also be excluded from our solution space which will be constructed for nonzero β_i-coefficients in Eq.(2.1).

- **The solutions;** $X_{1,\ldots,7}, K_{1,\ldots,14}$ **:**

In general, we will exclude the cases when $A = 0$ as they correspond to $\beta_3 = 0$. The cases when $B = 0$ with $C/A > 0$ may have the minimum polynomials either $m(X) = AX^2 + C\mathbf{1}_4$, or $m(X) = X(AX^2 + C\mathbf{1}_4)$. However, both cases

are restricted to $\beta_2 = 0$. Thus, we will exclude them. On the other hand, when $B = 0$, $C/A < 0$ we have the solutions $K_{1,\ldots,14}$. Here

$$e_1 = tr(K_j) = n\tilde{\lambda} = n\sqrt{-C/A}, \tag{3.79}$$

and

$$tr(K_j)^2 = m(\tilde{\lambda})^2 = \frac{m}{n^2}(e_1)^2, \tag{3.80}$$

where $(n, m) =(4, 4), (-4, 4), (2, 4), (0, 4), (-2, 4), (3, 3), (2, 2), (1, 1), (-3, 3), (-2, 2), (-1, 1), (0, 2), (1, 3), (-1, 3)$ for $K_{1,2,\ldots,13,14}$, respectively. For K_4, K_{12}, $e_1 = 0$, from the condition $B = 0$ we observe that for these cases to be solutions we must have $\beta_2 = 0$ thus, we exclude these solutions too. For the other cases from the above relations we have

$$e_2 = \frac{1}{2}\big((e_1)^2 - tr(K_j)^2\big) = \frac{n^2 - m}{2n^2}(e_1)^2. \tag{3.81}$$

Now, if we use Eqs. (2.17), and (3.81) in Eq.(3.79) we obtain the quadratic equation

$$\big(1 + \frac{n^2 - m}{2}\big)(e_1)^2 + \frac{n^2\beta_2}{\beta_3}e_1 + \frac{n^2\beta_1}{\beta_3} = 0. \tag{3.82}$$

When $n^2 - m \neq -2$ this equation has the solutions

$$e_1 = \frac{-\frac{n^2\beta_2}{\beta_3} \pm \sqrt{\frac{n^4(\beta_2)^2}{(\beta_3)^2} - 2\big(2 + n^2 - m\big)\frac{n^2\beta_1}{\beta_3}}}{2 + n^2 - m}. \tag{3.83}$$

We observe that if

$$\frac{n^2(\beta_2)^2}{(\beta_3)^2} - 2\big(2 + n^2 - m\big)\frac{\beta_1}{\beta_3} \geq 0, \tag{3.84}$$

e_1 is real and via the relation Eq.(3.79) the condition $C/A < 0$ is satisfied. For these solutions to exist we must also satisfy the condition $B = 0$, which reads

$$\big[n^4 - (m-2)^2\big](\beta_2)^2 - 2\big(2 + n^2 - m\big)n^2\beta_1\beta_3 = 0, \tag{3.85}$$

where we have made use of the Eqs.(2.17), and (3.83). For the solutions K_8, K_{11} (3.85) gives $\beta_1\beta_3 = 0$, which leads us to exclude these solutions. A close inspection denotes that when (3.85) is satisfied (3.84) is automatically satisfied, thus, Eq.(3.85) is a stronger condition to be considered uniquely.

Therefore, the condition (3.85) is the unique defining condition for the existence of the solutions $K_1, K_2, K_3, K_5, K_6, K_7, K_9, K_{10}$. On the other hand, when $n^2 - m = -2$, namely, for the cases K_{13}, K_{14} Eq.(3.82) leads us to the result

$$e_1 = -\frac{\beta_1}{\beta_2}. \tag{3.86}$$

e_2 must again be read from Eq.(3.81). In this case, since $\tilde{\lambda}$ must be positive, via Eq.(3.79) we see that; when $\beta_1/\beta_2 > 0$, the solution is K_{14}, and when $\beta_1/\beta_2 < 0$, the solution is K_{13}. Besides, since $C/A = -n^{-2}(e_1)^2 = -(e_1)^2$ the condition $C/A < 0$ is automatically satisfied for these solutions. The other restriction $B = \beta_2 + \beta_3 e_1 = 0$ becomes

$$(\beta_2)^2 - \beta_1\beta_3 = 0, \tag{3.87}$$

which is left as the unique defining condition for K_{13}, K_{14} to exist. For all these solutions discussed above, the matrix entries can be found as $\tilde{\lambda} = \sqrt{-C/A} = e_1/n$ by substituting the relevant e_1, and n values case by case. We should state however, that when one computes e_1 from Eq.(3.83), then substitutes the results in Eqs. (3.14), (3.16), (3.17) for the cases K_1, K_2, K_6, K_7, K_9, and K_{10}, respectively one finds that $K_1 = K_2$, $K_6 = K_9$, and $K_7 = K_{10}$. For this reason, we will define the matrices $K \equiv K_1 = K_2$, $K' \equiv K_6 = K_9$, and $K'' \equiv K_7 = K_{10}$ for the rest of our analysis. Now, we will turn our attention to the condition $C = 0$ which generates the $X-$series of solutions. Firstly we should note that when $C = 0$, in the previous subsection, the solutions coming from the minimum polynomials $m(X) = X$, and $m(X) = X^2$ are excluded as they lead to the restriction $\beta_1 = 0$. Besides, the solutions with $m(X) = AX^3$ when $B = C = 0$ are also disregarded as they are restricted to the cases $\beta_1 = \beta_2 = 0$. For X_1 we have $e_1 = trX_1 = -4B/A$ which gives

$$e_1 = -\frac{4\beta_2}{3\beta_3}. \tag{3.88}$$

We also get

$$e_2 = \frac{1}{2}\big((e_1)^2 - tr(X_1)^2\big) = \frac{3}{8}(e_1)^2. \tag{3.89}$$

By using these results, and by referring to Eq.(2.17) once more, the existence condition $C = 0$ becomes

$$2(\beta_2)^2 - 3\beta_1\beta_3 = 0. \tag{3.90}$$

Then, by making use of Eq.(3.88) we can also explicitly write X_1 as

$$X_1 = -\frac{\beta_2}{3\beta_3}\mathbf{1}_4. \tag{3.91}$$

For the other cases; $X_{2,3,4,5,6,7}$, $e_1 = trX_i = -nB/A$ with $n = 1$ for X_2, X_5, X_6, and $n = 2$ for X_3, X_7, and $n = 3$ for X_4, respectively. Thus, for these cases when $n \neq 1$ we obtain

$$e_1 = -\frac{n\beta_2}{(n-1)\beta_3}. \tag{3.92}$$

For $n = 1$, substitution of A, B via Eq.(2.17) in $e_1 = -nB/A$ gives $\beta_2 = 0$, hence, we exclude the solutions X_2, X_5, X_6. For the solutions $X_{3,4,7}$, we can compute e_2 as

$$e_2 = \frac{1}{2}\big((e_1)^2 - tr(X_i)^2\big) = \frac{n-1}{2n}(e_1)^2. \tag{3.93}$$

Similarly, for these cases the condition $C = 0$ reads

$$n(\beta_2)^2 - 2(n-1)\beta_1\beta_3 = 0. \tag{3.94}$$

Also, by using Eqs.(2.17), and (3.92) we can explicitly compute the matrix entries of $X_{3,4,7}$ as

$$-\frac{B}{A} = -\frac{\beta_2}{(n-1)\beta_3}. \tag{3.95}$$

4 The decoupling solution space

In the previous section, we have explicitly worked out and constructed the Jordan canonical forms of the nontrivial fixed-eigenvalue solutions of Eq.(3.1) in terms of the β_i-coefficients when neither of $\beta_{1,2,3}$ is chosen to be vanishing. Let us first define the set composed of these solutions

$$\begin{aligned}\mathcal{J} \equiv \Big\{&U_1, U_2, U_3, U_8, U_{11}, U, U', U'', V_0, V_1, V_2, V_3, V_4, V_5, V_6, V_{10}, Y, X_1,\\ &X_3, X_4, X_7, K, K', K'', K_3, K_5, K_{13}, K_{14}\Big\}.\end{aligned} \tag{4.1}$$

In the Appendix, we give a summary of the exact forms of these matrices whose entries are derived in terms of the β_i-coefficients in Section three. Next, we define the set of matrices

$$\mathcal{M} \equiv \Big\{M_J = M^{-1}JM \;\big|\; \forall J \in \mathcal{J},\ \text{and}\ \det M \neq 0\Big\}, \tag{4.2}$$

where M is any invertible real constant matrix. As we discussed before, the similarity equivalence classes of the elements of $\mathcal{J}$ are also solutions of Eq.(2.16). Bearing this in mind now, the set of matrix functions

$$\mathcal{S} \equiv \left\{ \sqrt{\Sigma} : \mathcal{U} \to \mathcal{M} \;\middle|\; \sqrt{\Sigma} \in C^{\infty} \right\}, \tag{4.3}$$

where $\mathcal{U}$ is a coordinate chart on the $4D$ spacetime manifold, is the complete set of general nontrivial solutions of Eq.(2.16) when $\beta_{1,2,3} \neq 0$. Of course, when the β_i-parameters are determined the appropriate elements of $\mathcal{J}$ must be chosen in other words, the ones whose domain of validity is satisfied by these parameters. We can also define the subset of $\mathcal{S}$

$$\mathcal{S} \supset \mathcal{P} \equiv \left\{ \sqrt{\Sigma} \equiv P_J = P^{-1}(x^{\mu}) J P(x^{\mu}) \;\middle|\; \forall J \in \mathcal{J}, \text{ and } \det P \neq 0 \right\}, \tag{4.4}$$

where $P(x^{\mu})$ is any smooth, invertible real matrix function on $\mathcal{U}$. The elements of $\mathcal{P}$ have fixed eigenvalues over the coordinate chart $\mathcal{U}$, and $\mathcal{P}$ is composed of the matrix functions whose ranges are grouped in the similarity equivalence classes of the elements of $\mathcal{J}$. Once more, when constructing $\mathcal{P}$ one must choose the elements of $\mathcal{J}$ which exist within the domain defined by the determined β_i-parameters, thus, one must respect the validity domains. We should remind the reader of the fact that the eigenvalues, e_1, and e_2 are the same for the elements of a similarity equivalence class. Referring to Eq.(2.4) we now have

$$f = g\Sigma. \tag{4.5}$$

In constructing the sets $\mathcal{S}$, and $\mathcal{P}$ we treated $\sqrt{\Sigma}$ as a general solution of the matrix equation Eq.(2.16) without considering its relevance to g, and f. However, Eq.(4.5) now brings a constraint on it. Since g, and f are symmetric $\sqrt{\Sigma}$ is restricted to the condition

$$g\Sigma = \Sigma^T g. \tag{4.6}$$

In other words, not all the elements of $\mathcal{S}$, and $\mathcal{P}$ will result in a symmetric f in Eq.(4.5) when a foreground metric g is specified. One, has to consider the subsets of them whose elements fall into the range of the product of two symmetric matrices when squared. For the elements of $\mathcal{P}$, Eq.(4.6) becomes

$$gP^{-1}J^2P = (P^{-1}J^2P)^T g. \tag{4.7}$$

When g is specified, one can choose ten function components of $P(x^\mu)$ freely, and determine the other six in terms of these, and the components of g by solving the six algebraic equations arising from Eq.(4.7). Then, one can explicitly construct the symmetric solution f via Eq.(4.5) in terms of g. Thus, we observe that we have $P = P[g, x^\mu]$ indeed. We should remark that, although this is the general solution construction method, one may prefer to choose simplified, or well-designed forms of P to generate solutions directly. For example, if one chooses $P = \mathbf{1}_4$, and take a diagonal g one immediately obtains diagonal f solutions for the choice of the diagonal elements of $\mathcal{J}$. We now define

$$\Gamma_{\mathcal{S}} = \big\{(g, f) \;\big|\; f = gX^2 \;\big| X \in \mathcal{S}, \text{ and } gX^2 = (X^T)^2 g\big\}, \tag{4.8}$$

and

$$\Gamma_{\mathcal{P}} = \big\{(g, f) \;\big|\; f = g(P_J)^2 \;\big| P_J \in \mathcal{P}, \text{ and } g(P_J)^2 = ((P_J)^T)^2 g\big\}, \tag{4.9}$$

in which, one must construct $P[g, x^\mu]$ in $P_J = P^{-1}JP$ such that it satisfies Eq(4.7) as we discussed above. Next, we will compute the effective cosmological constant in Eq.(2.15) which enters into the g-equation Eq.(2.6) as an effective presence of the background metric. Taking the trace of Eq.(2.16) leads us to

$$\beta_1 e_1 + 2\beta_2 e_2 + 3\beta_3 e_3 = 0, \tag{4.10}$$

where we have made use of Eqs.(2.3). By using this relation in Eq.(2.14), and by also referring to Eq.(2.2) we conclude that

$$\tilde{\Lambda} = -\frac{4}{3}\beta_1 e_1 - \frac{2}{3}\beta_2 e_2, \tag{4.11}$$

where for a particular solution of the form

$$f = g\big(P^{-1}[g, x^\mu] J P[g, x^\mu]\big)^2 = gP^{-1}[g, x^\mu] J^2 P[g, x^\mu], \tag{4.12}$$

with $J \in \mathcal{J}$ one must read the appropriate $e_1(\beta_i)$, and $e_2(\beta_i)$ for the particular choice of J from its parametric structure derived in the previous section. This is due to the fact that e_1, and e_2 are the symmetric polynomials of $\sqrt{\Sigma} = P^{-1}[g, x^\mu] J P[g, x^\mu]$, and under similarity transformations they remain the same so they are also the symmetric polynomials of the particular J which is chosen to generate the solution. Since, $\tilde{\Lambda}$ must stay constant over a coordinate chart $\mathcal{U}$ to be able to satisfy Eq.(2.13). We will define

$$\Gamma = \Gamma'_{\mathcal{S}} \cup \Gamma_{\mathcal{P}}, \tag{4.13}$$

as the general solution space of Eq.(2.12) where the set $\Gamma'_{\mathcal{S}}$ formally corresponds to the elements in the set $\Gamma_{\mathcal{S}} - \Gamma_{\mathcal{P}}$ with invariant e_1, e_2, or invariant $\tilde{\Lambda}$ over $\mathcal{U}$. We conclude that, the elements of Γ constitute the decoupling solution space of bigravity, with $\Gamma_{\mathcal{P}}$ being an analytically expressible subset of it that is composed of similarity equivalence classes of matrix functions.

5 Proportional Backgrounds

Before we conclude, in this section, we present a class of exact solutions in which the two metrics are proportional to each other so that

$$f_{\mu\nu} = \mathcal{C}^2 g_{\mu\nu}. \tag{5.1}$$

These solutions are generated by the elements of Eq. (4.1) which are proportional to the unit matrix, namely, by $\{U_1, U_2, V_0, K, X_1\}$. Among such solutions which are known as the proportional backgrounds in the literature [20], as a special class there exists the subset in which both of the metrics are diagonal in the same coordinate system. This is due to the fact that in the action Eq. (2.1) both metrics are defined on the same coordinate patch, and in our formalism of deriving these solutions we have kept the same coordinate chart for both of the metrics all through our analysis. After describing the general structure of the solutions in Eq. (5.1) we will explicitly construct examples of bidiagonal-metric solutions in which both of the metrics are of Friedmann-Lemaitre-Robertson-Walker (FLRW) type. The values of the proportionality constants $\mathcal{C}$ in Eq. (5.1) which only depend on the $\{\beta_i\}$-coupling constants for $\{U_1, U_2, V_0, X_1\}$ are given in the Eqs. (3.32), (3.43), (3.91), respectively. Also, we have $\mathcal{C} = \tilde{\lambda} = e_1/n$, with $n = 4, -4$ for K_1, K_2. Here, when we computed e_1 via Eq. (3.83) for $(n, m) = (4, 4), (-4, 4)$ for K_1, K_2, respectively we previously found that $K_1 = K_2$, and we have already defined the matrix $K \equiv K_1 = K_2$. Thus, one can take $\mathcal{C} = \tilde{\lambda} = e_1[K_1]/4$ for K. The proportionality constant $\mathcal{C}$ corresponding to K also depends only on the coefficients $\{\beta_i\}$. As we mentioned in Section two, since the configuration in Eq. (5.1) satisfies Eq. (2.12) the second of the Bianchi identities in Eq. (2.11) is directly fulfilled, and the first one leads to a contribution of a cosmological constant term to the g-metric equations Eq. (2.6). The value of this effective cosmological constant can be computed from Eq. (4.11) as

$$\tilde{\Lambda} = -\frac{16}{3}\beta_1 \mathcal{C} - 4\beta_2 \mathcal{C}^2. \tag{5.2}$$

On the other hand, for the solutions of type Eq. (5.1) since $\tau = 0$ we see via Eq. (2.9) that the f-metric equation Eq. (2.7) gets no contribution from the g-f interaction terms in the action Eq. (2.1). Therefore, for the proportional background solutions the g, and f-metric equations become

$$G^g_{\mu\nu} - \frac{1}{2}(\Lambda^g + m^2\tilde{\Lambda})g_{\mu\nu} = 8\pi G T^g_{M\,\mu\nu}, \tag{5.3}$$

and

$$\kappa\big[G^f_{\mu\nu} - \frac{1}{2}\Lambda^f f_{\mu\nu}\big] = \epsilon 8\pi G T^f_{M\,\mu\nu}, \tag{5.4}$$

respectively, where $G^g_{\mu\nu}$, and $G^f_{\mu\nu}$ are the corresponding Einstein tensors. Since, for the solutions Eq. (5.1) we have $(G^f)^\mu{}_\nu = (G^g)^\mu{}_\nu/\mathcal{C}^2$ if one chooses the fine-tuning of the g, and the f-sector sources as

$$\Lambda^f = \frac{1}{\mathcal{C}^2}(\Lambda^g + m^2\tilde{\Lambda}), \quad (T^f_M)^\mu{}_\nu = \frac{\kappa}{\epsilon\,\mathcal{C}^2}(T^g_M)^\mu{}_\nu, \tag{5.5}$$

then the equations (5.3), (5.4) become equivalent upon the choice of the solutions in Eq. (5.1). If one of these equations is satisfied the other is also automatically satisfied. The fine-tuning that is introduced above is a typical characteristic of the proportional backgrounds [20]. However, we should state that the proportional backgrounds derived here are new and different from the already-known ones in the literature. The reader may refer to [20] to observe that the known solutions of type (5.1) are derived by excluding the possibility of the vanishing of τ which results in a different fine-tuning condition that determines the permissible values of $\mathcal{C}$. Therefore, we conclude that for any solution of the Einstein equations Eq. (5.3) which are modified by an effective cosmological constant that is proportional to m^2 with the metric g, matter cosmological constant Λ^g, and the matter energy-momentum tensor $T^g_{M\,\mu\nu}$ the metric f chosen as in Eq (5.1) via the introduction of the f-cosmological constant and matter sources as in Eq. (5.5) is a solution of Eq. (5.4). Hence, consequently the metrics g, and $f = \mathcal{C}^2 g$ become the solutions of the bigravity action Eq (2.1). Next, we will construct an explicit example. Let us take the two proportional metrics as

FLRW type

$$g=-dt^2+\frac{a^2(t)}{1-kr^2}dr^2+a^2(t)r^2d\theta^2+a^2(t)r^2sin^2\theta d\varphi^2,$$

$$f=-\mathcal{C}^2dt^2+\frac{\mathcal{C}^2a^2(t)}{1-kr^2}dr^2+\mathcal{C}^2a^2(t)r^2d\theta^2+\mathcal{C}^2a^2(t)r^2sin^2\theta d\varphi^2, \tag{5.6}$$

with any of the proportionality constants $\mathcal{C}$ corresponding to the derived solutions $\{U_1, U_2, V_0, K, X_1\}$. We also take the $g-$matter energy-momentum tensor in perfect fluid form

$$T_M^g=\begin{pmatrix} -\rho(t) & 0 & 0 & 0 \\ 0 & p(t) & 0 & 0 \\ 0 & 0 & p(t) & 0 \\ 0 & 0 & 0 & p(t) \end{pmatrix}, \tag{5.7}$$

where we define the matrix $[T_M^g]^\mu{}_\nu \equiv (T_M^g)^\mu{}_\nu$. Substituting g from Eq. (5.6), and also Eq. (5.7) into Eq. (5.3) leads to the Friedmann equations

$$\left(\frac{\dot{a}}{a}\right)^2+\frac{k}{a^2}=\frac{8\pi G}{3}\rho-\frac{\Lambda^g+m^2\tilde{\Lambda}}{6}, \tag{5.8}$$

and

$$\frac{2\ddot{a}}{a}=-\left(\frac{\dot{a}}{a}\right)^2-\frac{k}{a^2}-8\pi Gp-\frac{\Lambda^g+m^2\tilde{\Lambda}}{2}. \tag{5.9}$$

We also have the fluid equation

$$\dot{\rho}=-\frac{3\dot{a}}{a}\big(p+\rho\big), \tag{5.10}$$

for the perfect fluid $g-$matter energy-momentum tensor in Eq. (5.7) coming from its conservation law $\nabla^\mu T^g_{M\,\mu\nu}=0$. As usual, if one specifies the equation of state $p=w\rho$ for the perfect fluid in Eq. (5.7) one can solve the scale factor $a(t)$ from the equations (5.8), (5.9), (5.10), and then determine the metric g in Eq. (5.6) explicitly. If furthermore, via the fine-tuning that is introduced in Eq. (5.5) we choose

$$T_M^f=\frac{\kappa}{\epsilon\,\mathcal{C}^2}\begin{pmatrix} -\rho(t) & 0 & 0 & 0 \\ 0 & p(t) & 0 & 0 \\ 0 & 0 & p(t) & 0 \\ 0 & 0 & 0 & p(t) \end{pmatrix}, \tag{5.11}$$

and $\Lambda^f = (\Lambda^g + m^2\tilde{\Lambda})/\mathcal{C}^2$ then the f metric in Eq. (5.6) automatically satisfies Eq. (5.4) since now this equation becomes equivalent to Eq. (5.3) as we discussed above. In this manner, the metrics in Eq. (5.6) form explicit, and exact background solutions of the bigravity action Eq. (2.1) for any value of $\mathcal{C}$ coefficients of the cases $\{U_1, U_2, V_0, K, X_1\}$. We should emphasize on an important point once more here, before we conclude. Both in our example of cosmological solutions, and in the more general proportional background cases given in Eq. (5.1), the g, and f metrics are defined in the same coordinate chart. These solutions are similar to the first class of solutions classified in [20]. For this reason, we do not have to solve a system of partial differential equations like the cases in the second class of solutions listed in [20] to find the Stückelberg fields which relate the g metric components to the f metric ones which are diagonal in different coordinate systems.

6 Concluding Remarks

Following the identification of a cubic matrix equation which serves as the dynamical decoupling solution ansatz for the two metrics of bigravity, we derived the general solutions of this non-constant-coefficient matrix equation. We first obtained the entire set of solutions of an ordinary cubic matrix equation by classifying them into similarity equivalence classes of diagonalizable and nondiagonalizable solutions. Later, starting from these cases we have derived the complete set of roots of the actual ansatz equation with coefficients also being functions of the elementary symmetric polynomials of the roots. Consequently, we obtained the Jordan canonical forms of all the equivalence classes of solutions written in terms of the β-coupling constants of the interaction term of the bigravity action. The decoupling ansatz contributes an effective cosmological constant to the foreground metric equation, while it leaves the background metric theory as a detached GR. By being able to express the elementary symmetric polynomials of the solutions of the cubic matrix ansatz equation in terms of the coupling constants of the theory we have also explicitly obtained the effective cosmological constant for each class of solutions. Later on, we presented a formal definition of the decoupling solution moduli of bigravity which forms an important branch of exact solutions of the theory. The analytical elements of this effective solution space can explicitly be constructed following the solution of six algebraic equations. In general, the decoupling solution space tells us which couple of

the metrics must be chosen to solve the independent Einstein equations in two decoupled GR-sectors (one having an always-nonvanishing cosmological constant due to the presence of the effective contribution coming from the interaction term). Finally, we presented a class of exact solutions in which the two metrics are proportional to each other in the same coordinate chart. We also explicitly constructed an example of the cosmological proportional background solutions of the massive bigravity. We should remark that the proportional background solutions obtained in this work are new and they differ from the already-existing ones in the literature [20] which are derived by excluding the possibility of the vanishing of the matrix τ which on the contrary sits at the focus of the present work.

In search for constructing self accelerating cosmologies, the cosmological solutions of the ghost-free bigravity have attracted a considerable attention in recent years [12, 13, 14, 17, 18, 19, 20, 21, 22, 23, 24]. The known class of solutions studied in the corresponding literature can be divided into three groups [20]; the class in which both metrics are proportional, a class of spherically symmetric solutions in which the background metric is nondiagonal, and solutions including diagonal but not proportional metrics. In this work, apart from the parametrically-reduced cases, we construct the general solution space of the bigravity field equations when the g, and the f-sectors completely decouple from each other and simply become GR with only an additional contribution of an effective cosmological constant to the foreground metric Einstein equations. Beside formally identifying the decoupling solution moduli of the theory, we have presented a complete set of similarity equivalence classes of analytic solutions up to solving a set of algebraic equations. Each of these classes contains functionally infinite degrees of freedom that generate proportional backgrounds, diagonal, and nondiagonal solutions. We should also emphasize that, clever choices of ansatz may substantially ease the general explicit-solution building method we discussed. We believe that, this complete classification of the decoupling solutions can be a good starting point to study the viable cosmological solutions of bigravity in a more systematical way.

A Appendix

In the Appendix, we present the collection of the matrices constituting the set $\mathcal{J}$ defined in Eq. (4.1). Their exact forms whose entries are functions of

the $\{\beta_i\}-$coefficients are derived in Section three by substituting the corresponding ansatz into the equation (3.1). Firstly,

$$U_1 = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}, U_2 = \begin{pmatrix} \lambda_2 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}, U_3 = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}, \tag{A.1}$$

where

$$\begin{aligned} \lambda_1 &= \frac{\beta_2 + \beta_3 c_1 - (\beta_3^2 c_1^2 + 4\beta_2\beta_3 c_1 + \beta_2^2 + 4\beta_1\beta_3)^{1/2}}{2\beta_3}, \\ \lambda_2 &= \frac{\beta_2 + \beta_3 c_1 + (\beta_3^2 c_1^2 + 4\beta_2\beta_3 c_1 + \beta_2^2 + 4\beta_1\beta_3)^{1/2}}{2\beta_3}, \end{aligned} \tag{A.2}$$

with

$$c_1 = \frac{-6\beta_2\beta_3 \mp 2(9\beta_2^2\beta_3^2 - 12\beta_1\beta_3^3)^{1/2}}{3\beta_3^2}. \tag{A.3}$$

On the other hand,

$$U_8 = \begin{pmatrix} \lambda_1' & 0 & 0 & 0 \\ 0 & \lambda_1' & 0 & 0 \\ 0 & 0 & \lambda_1' & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad U_{11} = \begin{pmatrix} \lambda_2' & 0 & 0 & 0 \\ 0 & \lambda_2' & 0 & 0 \\ 0 & 0 & \lambda_2' & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A.4}$$

where

$$\begin{aligned} \lambda_1' &= \frac{\beta_2 + \beta_3 c_2 - ((\beta_2 + \beta_3 c_2)^2 + 8\beta_1\beta_3 + 4\beta_2\beta_3 c_2)^{1/2}}{2\beta_3}, \\ \lambda_2' &= \frac{\beta_2 + \beta_3 c_2 + ((\beta_2 + \beta_3 c_2)^2 + 8\beta_1\beta_3 + 4\beta_2\beta_3 c_2)^{1/2}}{2\beta_3}, \end{aligned} \tag{A.5}$$

with

$$c_2 = \frac{-3\beta_2\beta_3 \pm 3(\beta_2^2\beta_3^2 - \beta_1\beta_3^3)^{1/2}}{\beta_3^2}. \tag{A.6}$$

Also,

$$U = \begin{pmatrix} -\frac{\beta_2}{2\beta_3} & 0 & 0 & 0 \\ 0 & -\frac{\beta_2}{2\beta_3} & 0 & 0 \\ 0 & 0 & -\frac{\beta_2}{2\beta_3} & 0 \\ 0 & 0 & 0 & \frac{3\beta_2 + 2\beta_3 e_1}{2\beta_3} \end{pmatrix}, \tag{A.7}$$

where e_1 is a free parameter such that $e_1 \in \left\{\mathcal{R} - \{-2\beta_2/\beta_3\}\right\}$.

$$U' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{\beta_2}{\beta_3} & 0 & 0 \\ 0 & 0 & -\frac{\beta_2}{\beta_3} & 0 \\ 0 & 0 & 0 & \frac{2\beta_2+\beta_3 e_1'}{\beta_3} \end{pmatrix}, \tag{A.8}$$

where e_1' is a free parameter such that $e_1' \in \left\{\mathcal{R} - \{-3\beta_2/\beta_3\}\right\}$. In addition,

$$U'' = \begin{pmatrix} -\frac{\beta_1}{\beta_2} & 0 & 0 & 0 \\ 0 & -\frac{\beta_1}{\beta_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A.9}$$

For the V-series we have,

$$V_0 = \begin{pmatrix} -\frac{\beta_2}{2\beta_3} & 0 & 0 & 0 \\ 0 & -\frac{\beta_2}{2\beta_3} & 0 & 0 \\ 0 & 0 & -\frac{\beta_2}{2\beta_3} & 0 \\ 0 & 0 & 0 & -\frac{\beta_2}{2\beta_3} \end{pmatrix}, V_1 = \begin{pmatrix} -\frac{\beta_2}{2\beta_3} & 1 & 0 & 0 \\ 0 & -\frac{\beta_2}{2\beta_3} & 0 & 0 \\ 0 & 0 & -\frac{\beta_2}{2\beta_3} & 0 \\ 0 & 0 & 0 & -\frac{\beta_2}{2\beta_3} \end{pmatrix},$$

$$V_2 = \begin{pmatrix} -\frac{\beta_2}{2\beta_3} & 0 & 0 & 0 \\ 0 & -\frac{\beta_2}{2\beta_3} & 1 & 0 \\ 0 & 0 & -\frac{\beta_2}{2\beta_3} & 0 \\ 0 & 0 & 0 & -\frac{\beta_2}{2\beta_3} \end{pmatrix}, V_3 = \begin{pmatrix} -\frac{\beta_2}{2\beta_3} & 0 & 0 & 0 \\ 0 & -\frac{\beta_2}{2\beta_3} & 0 & 0 \\ 0 & 0 & -\frac{\beta_2}{2\beta_3} & 1 \\ 0 & 0 & 0 & -\frac{\beta_2}{2\beta_3} \end{pmatrix},$$

$$V_4 = \begin{pmatrix} -\frac{\beta_2}{2\beta_3} & 1 & 0 & 0 \\ 0 & -\frac{\beta_2}{2\beta_3} & 0 & 0 \\ 0 & 0 & -\frac{\beta_2}{2\beta_3} & 1 \\ 0 & 0 & 0 & -\frac{\beta_2}{2\beta_3} \end{pmatrix}, V_5 = \begin{pmatrix} -\frac{\beta_2}{\beta_3} & 1 & 0 & 0 \\ 0 & -\frac{\beta_2}{\beta_3} & 0 & 0 \\ 0 & 0 & -\frac{\beta_2}{\beta_3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$V_6 = \begin{pmatrix} -\frac{\beta_2}{\beta_3} & 0 & 0 & 0 \\ 0 & -\frac{\beta_2}{\beta_3} & 1 & 0 \\ 0 & 0 & -\frac{\beta_2}{\beta_3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad V_{10} = \begin{pmatrix} -\frac{\beta_2}{\beta_3} & 0 & 0 & 0 \\ 0 & -\frac{\beta_2}{\beta_3} & 0 & 0 \\ 0 & 0 & -\frac{\beta_2}{\beta_3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A.10}$$

Now,

$$Y = \begin{pmatrix} R & I & 0 & 0 \\ -I & R & 0 & 0 \\ 0 & 0 & R & I \\ 0 & 0 & -I & R \end{pmatrix}, \tag{A.11}$$

where

$$R = -\frac{\beta_2}{2\beta_3}, \quad I = -\frac{(3\beta_2^2 - 4\beta_1\beta_3)^{1/2}}{2\beta_3}. \tag{A.12}$$

The X-series elements in $\mathcal{J}$ are

$$X_1 = \begin{pmatrix} -\frac{\beta_2}{3\beta_3} & 0 & 0 & 0 \\ 0 & -\frac{\beta_2}{3\beta_3} & 0 & 0 \\ 0 & 0 & -\frac{\beta_2}{3\beta_3} & 0 \\ 0 & 0 & 0 & -\frac{\beta_2}{3\beta_3} \end{pmatrix}, X_3 = \begin{pmatrix} -\frac{\beta_2}{\beta_3} & 0 & 0 & 0 \\ 0 & -\frac{\beta_2}{\beta_3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_4 = \begin{pmatrix} -\frac{\beta_2}{2\beta_3} & 0 & 0 & 0 \\ 0 & -\frac{\beta_2}{2\beta_3} & 0 & 0 \\ 0 & 0 & -\frac{\beta_2}{2\beta_3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad X_7 = \begin{pmatrix} -\frac{\beta_2}{\beta_3} & 0 & 0 & 0 \\ 0 & -\frac{\beta_2}{\beta_3} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A.13}$$

If we turn our attention on the K-series we have

$$K = \begin{pmatrix} \tilde{\lambda} & 0 & 0 & 0 \\ 0 & \tilde{\lambda} & 0 & 0 \\ 0 & 0 & \tilde{\lambda} & 0 \\ 0 & 0 & 0 & \tilde{\lambda} \end{pmatrix}, \tag{A.14}$$

with

$$\tilde{\lambda} = -\frac{2}{7}\frac{\beta_2}{\beta_3} \pm \frac{1}{7}\left(4\frac{\beta_2^2}{\beta_3^2} - 7\frac{\beta_1}{\beta_3}\right)^{1/2}. \tag{A.15}$$

In addition, we also have

$$K_3 = \begin{pmatrix} c_3 & 0 & 0 & 0 \\ 0 & c_3 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & -c_3 \end{pmatrix}, K_5 = \begin{pmatrix} -c_3 & 0 & 0 & 0 \\ 0 & c_3 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & c_3 \end{pmatrix}, K' = \begin{pmatrix} c_4 & 0 & 0 & 0 \\ 0 & c_4 & 0 & 0 \\ 0 & 0 & c_4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K'' = \begin{pmatrix} c_5 & 0 & 0 & 0 \\ 0 & c_5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_{13} = \begin{pmatrix} c_6 & 0 & 0 & 0 \\ 0 & c_6 & 0 & 0 \\ 0 & 0 & -c_6 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K_{14} = \begin{pmatrix} c_7 & 0 & 0 & 0 \\ 0 & -c_7 & 0 & 0 \\ 0 & 0 & -c_7 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A.16}$$

where

$$
\begin{aligned}
c_3 &= -\frac{\beta_2}{\beta_3} \pm \left(\frac{\beta_2^2}{\beta_3^2} - \frac{\beta_1}{\beta_3}\right)^{1/2}, \quad c_4 = -\frac{3}{8}\frac{\beta_2}{\beta_3} \pm \frac{1}{8}\left(9\frac{\beta_2^2}{\beta_3^2} - 16\frac{\beta_1}{\beta_3}\right)^{1/2}, \\
c_5 &= -\frac{1}{2}\frac{\beta_2}{\beta_3} \pm \frac{1}{2}\left(\frac{\beta_2^2}{\beta_3^2} - 2\frac{\beta_1}{\beta_3}\right)^{1/2}, \quad c_6 = -\frac{\beta_1}{\beta_2}, \quad c_7 = \frac{\beta_1}{\beta_2}.
\end{aligned}
\tag{A.17}
$$

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